➔ RIDDLE #7

ESA's NEO Coordination Centre

Help Santa to save the world!

Yesterday ESA's Flyeye telescope detected a 200 meter sized asteroid that is on a collision course with the Earth. (This is fictitious!!) And not like in the riddle of last month, where the asteroid was approaching quite slowly and had a possible impact in 2049. No! This time the asteroid is coming on a retrograde orbit (inclination of 180 degrees) with an aphelion at 20 au and a perihelion of 1 au, with a predicted impact velocity of 71 km/s, pretty similar to the orbits of the Leonid meteoroids. And the predicted impact on Earth is on 24 December 2022. We have just 2 years to react.

What are our options? Let's look at Figure 1 where the Planetary Defenders have designed mitigation scenarios as function of warning time and size of the approaching asteroid. For a gravity tractor it is clearly too late, nuclear detonations are too risky and also not needed, but for civil defence (like evacuation) the asteroid is too big.

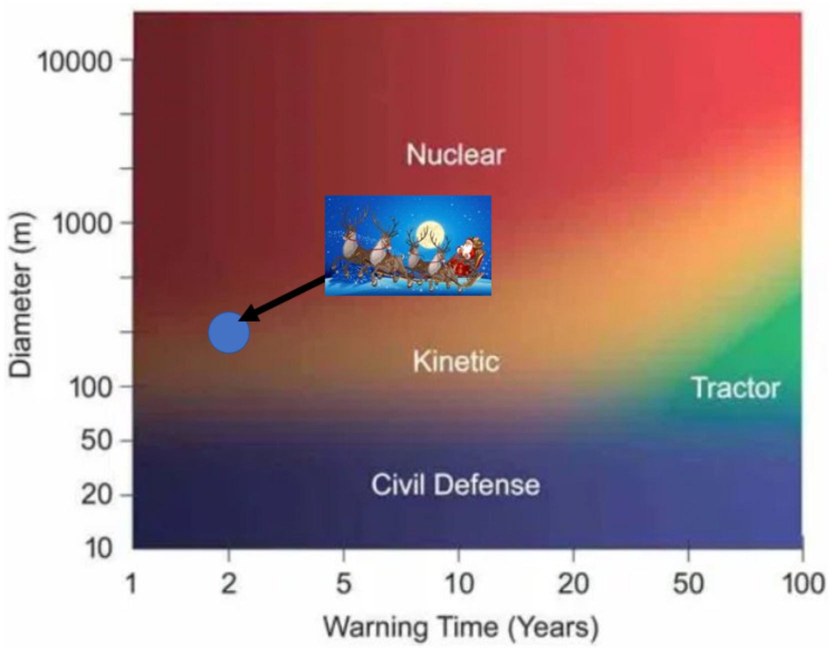

Figure 1: Planetary defence options as function of warning time and size of the asteroid. Image Courtesy of Tim Warchocki.

We need a kinetic impactor to deflect the asteroid. Unfortunately, ESA, NASA and all other space-faring nations are not yet ready to launch in such a short time a deflection mission. But Christmas time is coming up and Santa Claus has already his sledges in low-Earth orbit, ready to be loaded with the presents for all 120 million children living in Europe.

The only way to save the Earth is to load a fraction of the presents into an extra sledge and slam this sledge into the asteroid to deflect it from its fatal trajectory.

And this is the Christmas riddle: How many grams needs each of the 120 million presents be reduced if this mass is used for the required impactor? Assume a launch of the Santa Claus sledge on 24 Dec 2020 into exactly the same orbit as the asteroid, but prograde (perihelion = 1 au, apohelion = 20 au, inclination = 0 deg) and ignore the dry mass of the sledge. Assume further a circular Earth orbit with a semi-major axis of 1 au, a specific density of the asteroid of 2.0 t/m³, and assume a central impact with impulse conservation, i.e. the energy of the impactor is transferred 100 % to the asteroid. The effect of the impact is such that the new perihelion of the asteroid will be 15 000 km lower as before the impact.

Answer

Let us first calculate the semi-major axis a, eccentricity e and orbital period P of the orbit of the asteroid and of the sledge of Santa, which are the same, just the inclination is different.

Semi-major axis	Eccentricity	Orbital period
10.5 au	0.90476	34.0 years

The impact will happen exactly after 1 year. Therefore the mean anomaly M in the orbit will be $\frac{2\pi}{P} = 10.6°$. With an iterative algorithm we can determine the eccentric anomaly $E = 50.7°$ and with

$$\tan^2\frac{f}{2} = \frac{1+e}{1-e}\tan^2\frac{E}{2}$$

we calculate the true anomaly $f = 129.5°$. And from

$$r = \frac{a(1-e^2)}{1+e\cos f}$$

we can deduce the heliocentric distance r at which the impact will happen. From the vis-viva equation

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \tag{1}$$

we derive the instantaneous velocity v of both asteroid and sledge, however they are in opposite directions.

We need to calculate the reduction of v such that the new perihelion is 15000 km smaller than before. Let us define α as

$$\alpha = \frac{v - \Delta V}{v}$$

(e.g. $\alpha = 0.9$ means a reduction of the velocity of 10%.)

With a reduced velocity of $v^+ = \alpha v$ we get via the vis-viva equation the new semi-major axis a^+ of the asteroid orbit after the impact:

$$a^+ = \frac{\mu r}{2\mu - {v^+}^2 r}$$

To get the new eccentricity, we will calculate the new semi-latus rectum p^+ via the equation for the transverse component of the velocity:

$$v_t^+ = \frac{\sqrt{\mu p^+}}{r}$$

with the semi-latus rectum $p^+ = a^+(1 - e^{+2})$.

We know that $v_t^+ = \alpha v_t$ and therefore

$$p^+ = \frac{(v_t^+ r)^2}{\mu}$$

This gives the new eccentricity:

$$e^+ = \sqrt{1 - \frac{p^+}{a^+}}$$

and the new perihelion radius:

$$r_p^+ = a^+(1 - e^+) \tag{2}$$

Inserting the previous equations into equation 2 yields this expression:

$$r_p^+ = \frac{\mu r}{2\mu - \alpha^2 v^2 r}\left(1 - \sqrt{1 - \frac{(\alpha v_t r)^2}{\mu(\frac{\mu r}{2\mu - \alpha^2 v^2 r})}}\right) \tag{3}$$

It is easy to write a small computer program to find the α which will provide the requested r_p^+, however, that would be too easy. Instead, we rearrange equation 3 such that we can calculate α analytically. After some manipulations equation 3 is transformed into

$$\alpha^4\left(\frac{v^2}{\mu}(v^2 r_p^{+2} - v_t^2 r^2)\right) + \alpha^2\left(\frac{2}{\mu r}(r r_p^+ v^2 - 2 r_p^{+2} v^2 + v_t^2 r^2)\right) + \frac{4 r_p^+}{r}\left(\frac{r_p^+}{r} - 1\right) = 0 \tag{4}$$

The solution of this equation which we are interested in is:

$$\alpha^2 = 2\frac{r - r_p^+}{r}\frac{\mu r_p^+}{r^2 v_t^2 - r_p^{+2} v^2}$$

Inserting the values for r, r_p^+, v and v_t gives $\alpha = 0.99994$ or a ΔV of 1.0 m/s.

Now we need to calculate the required mass of the sledge to achieve this velocity change. Conservation of momentum means (m_A is the mass of the asteroid: 8.4 million tons, m_S is the mass of Santa's sledge):

$$m_A v - m_S v = (m_A + m_S)\alpha v$$

which gives $m_S = m_A \frac{1-\alpha}{1+\alpha}$

For the calculated α the results is $m_S = 245$ tons. Distributed over 120 million children this is just a mass of 2 gram, equivalent to less than one chocolate bonbon.

Correct responses

This time we had one correct answer by:

- Tony Evans

Congratulations!