→ RIDDLE #6

ESA's NEO Coordination Centre

Earth and Mars Impactor

Every year, around 18 November, the Earth travels through a cloud of cometary dust and we can witness another display of the Leonid meteor shower. The meteors are created when the dust ejected many decades or centuries ago by the comet Tempel-Tuttle enters in the dense atmosphere of the Earth and disintegrates at altitudes around 80 to 100 km. Typically the sizes of the dust particles are just a few millimetres.

Comet Tempel-Tuttle is in a retrograde orbit with a period of 33 years. Its aphelion is close to the orbit of Uranus at around 20 au. And the dust particles are in very similar orbits. Therefore their velocity when they enter the Earth atmosphere is very high (72 km/s). "Average" meteors in prograde orbits have typical impact velocities around 20 km/s. And this brings us to the riddle of this month:

The Flyeye telescope has detected a 500 m big NEO. (This is fictitious!!) After 3 days the NEOCC publishes some more details. The inclination is almost 0 degrees, the impact probability on 22 November 2049 is 1:230 000 and the impact velocity on the Earth surface is 20 km/s (ignoring atmospheric effects). A day later Juan Luis Cano from NEOCC adds that an impact on Mars can also not be excluded. And by chance the impact velocity on Mars would also be 20 km/s.

What is the perihelion of this peculiar NEO?
N.B.: for simplification, assume circular orbits for Earth and Mars, with Mars orbiting in the ecliptic plane. Fig. 1 shows the orbit of an object that is crossing Earth and Mars orbit.

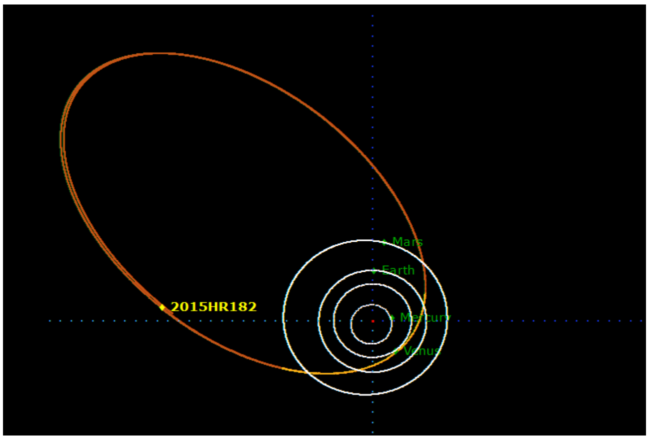


Figure 1: The orbit of asteroid 2015 HR182 is crossing both Earth and Mars orbit.

Answer

Let us first agree on the physical parameters that we are going to use to solve this riddle. We need the gravitational constants of the Earth, Mars and Sun and the radius and semi-major axis of the orbits of Earth and Mars (we assume perfect circular and co-planar orbits):

	Earth	Mars	Sun
Gravitational constant μ (km^3/s^2)	398602	42828	$1.327124 \cdot 10^{11}$
Radius (km)	6378	3397	
Semi-major axis (km)	$149.6 \cdot 10^6$	$227.9 \cdot 10^6$	

The impact velocity on the planet surface is given by:

$$v_i = \sqrt{\frac{2\mu}{R_{planet}} + v_\infty^2} \tag{1}$$

From this equation we can derive the required approach velocity at "infinity" by inverting the equation and inserting 20 km/s for v_i. $\sqrt{2\mu/R}$ is the so called escape velocity of a planet. For Earth and Mars, escape velocity, the required approach velocity and the orbital velocity are given in this table:

	Earth	Mars
Escape velocity (km/s)	11.2	5.0
Approach velocity v_∞ (km/s)	16.6	19.4
Orbital velocity (km/s)	29.8	24.1

If we decompose the velocity of the asteroid in a radial component v_r and a transverse component v_t then this relation holds:

$$v_\infty^2 = (v_t - v_{Planet})^2 + v_r^2 \tag{2}$$

Now we apply the vis-viva equation for the asteroid at the point of impact (but ignoring the gravity of the planet):

$$v_{NEO}^2 = v_r^2 + v_t^2 = \mu_{Sun}\left(\frac{2}{r} - \frac{1}{a}\right) \tag{3}$$

From equations 2 and 3, we derive this equation for v_t:

$$v_t = \frac{1}{2v_{Planet}}\left[\mu_{Sun}\left(\frac{2}{r} - \frac{1}{a}\right) + v_{Planet}^2 - v_\infty^2\right] \tag{4}$$

But we know that

$$v_t = \frac{\sqrt{\mu p}}{r}$$

with the semi-latus rectum $p = a(1 - e^2)$.

We enter the values for Mars into equation 4 and solve for the v_∞^2 at Mars:

$$v_\infty^{Mars^2} = \mu_{Sun}\left(\frac{2}{a_{Mars}} - \frac{1}{a_{NEO}}\right) + v_{Mars}^2 - 2v_{Mars}\frac{\sqrt{\mu p}}{a_{Mars}} \tag{5}$$

Next we calculate $\sqrt{\mu p} = r \cdot v_t$ at the Earth with v_t from equation 4 and insert this into equation 5. This yields:

$$
\begin{aligned}
v_{\infty}^{\mathrm{Mars}^2} = {} & \mu_{\mathrm{Sun}}\left(\frac{2}{a_{\mathrm{Mars}}}-\frac{1}{a_{\mathrm{NEO}}}\right)+v_{\mathrm{Mars}}^2-\frac{v_{\mathrm{Mars}}}{v_{\mathrm{Earth}}}\frac{a_{\mathrm{Earth}}}{a_{\mathrm{Mars}}} \\
& \left[\mu_{\mathrm{Sun}}\left(\frac{2}{a_{\mathrm{Earth}}}-\frac{1}{a_{\mathrm{NEO}}}\right)+v_{\mathrm{Earth}}^2-v_{\infty}^{\mathrm{Earth}^2}\right]
\end{aligned}
\tag{6}
$$

All parameters in equation 6 are known except a_{NEO}. Hence we can solve the equation:

$$
a_{\mathrm{NEO}} = \frac{\mu_{\mathrm{Sun}}(1-\mathrm{ratio})}{v_{\mathrm{Mars}}^2 - v_{\infty}^{\mathrm{Mars}^2} + \frac{2\mu_{\mathrm{Sun}}}{a_{\mathrm{Mars}}} - \mathrm{ratio}\left(3v_{\mathrm{Earth}}^2 - v_{\infty}^{\mathrm{Earth}^2}\right)}
\tag{7}
$$

with ratio $= \frac{a_{\mathrm{Earth}}}{a_{\mathrm{Mars}}} \cdot \frac{v_{\mathrm{Mars}}}{v_{\mathrm{Earth}}}$.

Inserting the parameters from the two tables above gives a semi-major axis of 4.0 au. Entering this value in equation 5, we can calculate the semi-latus rectum p and with this the eccentricity and the perihelion of the NEO.

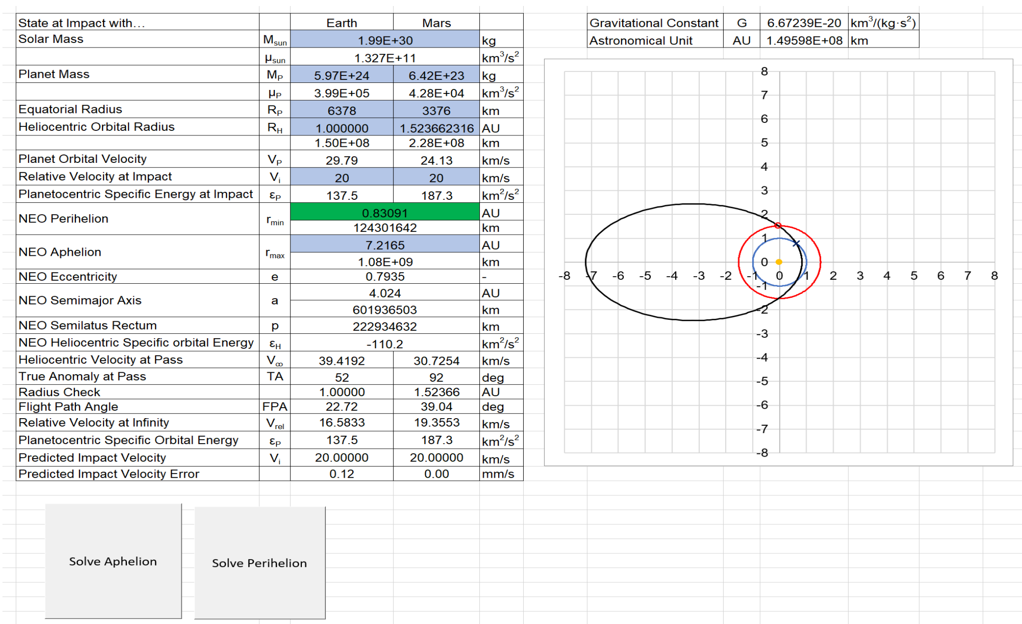
The eccentricity is 0.794 and the perihelion is 0.83 au.

Correct responses

This time we had two correct answers by:

- Max Fagin
- Julian Eslinger

Both found the solutions in different ways. Julian modelled the trajectory like a interplanetary mission with given flyby constraints, whereas Max demonstrated that the riddle could also be solved with an Excel spreadsheet which included this nice illustration:



State at Impact with…		Earth	Mars	
Solar Mass	M_{sun}	1.99E+30		kg
	μ_{sun}	1.327E+11		km^3/s^2
Planet Mass	M_P	5.97E+24	6.42E+23	kg
	μ_P	3.99E+05	4.28E+04	km^3/s^2
Equatorial Radius	R_P	6378	3376	km
Heliocentric Orbital Radius	R_H	1.000000	1.523662316	AU
		1.50E+08	2.28E+08	km
Planet Orbital Velocity	V_P	29.79	24.13	km/s
Relative Velocity at Impact	V_i	20	20	km/s
Planetocentric Specific Energy at Impact	ε_P	137.5	187.3	km^2/s^2
NEO Perihelion	r_{min}	0.83091		AU
		124301642		km
NEO Aphelion	r_{max}	7.2165		AU
		1.08E+09		km
NEO Eccentricity	e	0.7935		-
NEO Semimajor Axis	a	4.024		AU
		601936503		km
NEO Semilatus Rectum	p	222934632		km
NEO Heliocentric Specific orbital Energy	ε_H	-110.2		km^2/s^2
Heliocentric Velocity at Pass	V_∞	39.4192	30.7254	km/s
True Anomaly at Pass	TA	52	92	deg
Radius Check		1.00000	1.52366	AU
Flight Path Angle	FPA	22.72	39.04	deg
Relative Velocity at Infinity	V_{rel}	16.5833	19.3553	km/s
Planetocentric Specific Orbital Energy	ε_P	137.5	187.3	km^2/s^2
Predicted Impact Velocity	V_i	20.00000	20.00000	km/s
Predicted Impact Velocity Error		0.12	0.00	mm/s

Gravitational Constant	G	6.67239E-20	$km^3/(kg \cdot s^2)$
Astronomical Unit	AU	1.49598E+08	km

Congratulations to the two winners and stay tuned for our Christmas riddle!