→ RIDDLE #1

ESA's NEO Coordination Centre

An NEO with a peculiar orbit

The recently discovered asteroid 2020 HY5, firstly observed by Mt. Lemmon Survey on 23 April 2020, has an interesting particularity: it roughly passes half of its orbital period of about 387 days inside 1.3 au and the other half outside. Such distance is used for the definition of NEOs: the perihelion distance of an NEO must be below 1.3 au. 2020 HY5 actually spends 192.0 days below 1.3 au and 194.9 days above that distance.

And here is a riddle:

- Assuming an NEO that spends exactly 50% of its time inside 1.3 au and 50% of its time outside 1.3 au, what would be the maximum aphelion such an NEO could have?
- As a bonus, would you be able to find similar cases in our database? (Hint: you can use the advanced search functionality in our left menu)

Please, send your responses before the proposed deadline to the following e-mail: neocc@ssa.esa.int.
Use as subject of your e-mail: "Riddle #1 – solution".
Moreover, please let us know if you would prefer not to have your name included in the list of correct replies.

Answer

For any given eccentricity e there is one semi-major axis a which satisfies the condition that the NEO spends 50% inside 1.3 au.
From $r = a\,\frac{1-e^2}{1+e\cos\theta}$ and r = 1.3 au we deduce the relation:

$$a = 1.3\,\frac{1+e\cos\theta}{1-e^2}$$

where $\theta(e)$ is the true anomaly when the NEO reaches 1.3 au (θ depends on e). The aphelion will be:

(1) $$r_a(e) = 1.3\,\frac{1+e\cos\theta}{1-e}$$

A very simple approach would be to code a function for $r_a(e)$ and determine the maximum of the function. The true anomaly θ is obtained via the eccentric anomaly E, which itself is obtained via the mean anomaly M which is $\pi/2$ (note: half of the time, hence $\pi/2$):

$$M = \frac{\pi}{2} = E - e\sin E$$

and

(2) $$\tan^2\frac{\theta}{2} = \frac{1+e}{1-e}\tan^2\frac{E}{2}$$

But this is not an elegant solution. Let's do it more mathematically.

To maximise r_a it is necessary that $\frac{dr_a}{de} = 0$. This gives:

$$0 = 1.3\,\frac{(1+e\cos\theta)'(1-e)+1+e\cos\theta}{(1-e)^2}$$

$$= 1.3\,\frac{\left(\cos\theta - e\sin\theta\,\frac{d\theta}{de}\right)\cdot\left(1-e\right)+1+e\cos\theta}{(1-e)^2}$$

which is equivalent to:

$$0 = 1 + \cos\theta - e\,(1-e)\sin\theta\,\frac{d\theta}{de}$$

So we need $\frac{d\theta}{de}$ which is equal to $\frac{d\theta}{dE}\cdot\frac{dE}{de}$.

The second part $\frac{dE}{de}$ we get quickly from deriving $\frac{\pi}{2} = E - e\sin E$ which yields:

(3) $$\frac{dE}{de} = \frac{\sin E}{1-e\cos E}$$

The first part $\frac{d\theta}{dE}$ is obtained from deriving eq. (2) on both sides. This gives:

$$\frac{1}{\cos^2\frac{\theta}{2}}\,d\theta = \left[\left(\frac{1+e}{1-e}\right)^{-1/2}\frac{2}{(1-e)^2}\frac{de}{dE}\tan\frac{E}{2} + \frac{\left(\frac{1+e}{1-e}\right)^{1/2}}{\cos^2\frac{E}{2}}\right]dE$$

which gives, after few manipulations:

$$d\theta = \left[\frac{\sin\theta}{1-e^2}\frac{de}{dE} + \frac{\left(\frac{1+e}{1-e}\right)^{1/2}\left(1+\cos\theta\right)}{1+\cos E}\right]dE$$

Now we can put everything together in this equation:

$$\frac{dr_a}{de} = 1.3\,\frac{\left(\cos\theta - e\sin\theta\,\frac{d\theta}{de}\right)\cdot\left(1-e\right)+1+e\cos\theta}{(1-e)^2}$$

After extensive manipulations this yields:

$$\frac{dr_a}{de} = 1.3\,\frac{1+e\cos\theta}{(1-e)(1-e^2)}\left[1+\cos\theta-(1-e^2)^{1/2}e\sin\theta\,\frac{dE}{de}\right]$$

with $\frac{dE}{de}$ as given in eq. (3). In order to vanish the part in the square brackets must be zero:

(4) $$0 = 1+\cos\theta-(1-e^2)^{1/2}e\sin\theta\,\frac{dE}{de}$$

A Python program was written to solve this equation (see Appendix below), which gives an eccentricity e = 0.552 and a true anomaly θ = 144.2 degrees. Entering these two values in eq. (1) yields the solution:

$$r_a = 1.603\text{ au}$$

Objects in NEOCC database

The objects in our database that best approach the obtained maximum are:

Designator	Perihelion (au)	Aphelion (au)	Time below 1.3 au (%)
2015XG55	0.4552	1.6040	49.93%
2017HF1	0.3513	1.603	49.77%
144900 2004VG64	0.3335	1.6031	49.71%
488474 1999HD1	0.6588	1.6039	49.20%

The list of the ten objects in our database that best approach to equal time below and above 1.3 au is the following:

Designator	Perihelion (au)	Aphelion (au)	Time below 1.3 au (%)
2020 HK	0.8523	1.5538	50.00%
2018 FJ2	1.1052	1.4490	50.01%
2016 LH10	0.8151	1.5633	50.01%
(537395) 2015 LG2	1.0307	1.4881	49.99%
2019 GV20	1.0645	1.4714	49.99%
2016 WD7	0.9904	1.5059	50.01%
2018 WH	0.8769	1.5465	50.01%
2015 RH2	0.8929	1.5414	50.03%
2016 FH14	1.0224	1.4923	49.96%
2009 FX10	0.8439	1.5555	50.04%

Correct responses

And this is the list of persons that have responded correctly to this riddle, in order of reception date:

- Tony Evans
- Oscar Fuentes Muñoz
- Daniel Estévez
- Mauro Pirarba
- Juan Luis Cano Rodríguez
- M. L. Brown

Particular mention is given to Daniel Estévez and Juan Luis Cano Rodríguez for the very high quality of the responses that they provided.

Congratulations to all for the great work done and thank you so much for participating in this riddle.

Appendix

Python code to solve eq. (4).

```
# An NEO is 50% of its time outside 1.3 AU and 50% inside. What is its maximum aphelion?

from numpy import pi, sin, cos, tan, arctan, sqrt
from scipy.optimize import fsolve

def derive_ra(ecc):
   """
   Description
   :param ecc: eccentricity of the NEO orbit
   :return: dra_de: derivative of r_a with respect to ecc
   """
   global theta
   #
   # iteration to convert mean anomaly of 90 deg to eccentric anomaly
   ecc_ano_old = 100
   ecc_ano     = pi / 2
   #
   while abs(ecc_ano - ecc_ano_old) > 1.e-6:
      ecc_ano_old = ecc_ano
      ecc_ano     = pi / 2 + ecc * sin(ecc_ano_old)
   #
   # derivative of eccentric anomaly with respect to eccentricity
   dE_de = sin(ecc_ano) / (1 - ecc * cos(ecc_ano))
   #
   # tan2fhalf is tangent squared of theta/2
   tan2fhalf = (1 + ecc) / (1 - ecc) * tan(ecc_ano / 2) ** 2
   #
   # theta is the true anomaly
   theta = arctan(sqrt(tan2fhalf)) * 2
   #
   # derivative of aphelion radius with respect to eccentricity (which must be 0)
   dra_de = 1 + cos(theta) - sqrt(1 - ecc * ecc) * ecc * sin(theta) * dE_de
   #
   return dra_de

R50        = 1.3     # radius outside which the NEO spends 50% of its time [AU]
ECC_guess  = 0.4
ECC        = fsolve(derive_ra, ECC_guess)
RA_MAX     = R50 * (1 + ECC * cos(theta)) / (1 - ECC)      # this is the maximum aphelion
SEMI_MAJOR = RA_MAX / (1 + ECC)
RP         = SEMI_MAJOR * (1 - ECC)

print("Semi-major axis, eccentricity, perihelion and max aphelion:",\
      SEMI_MAJOR, ECC, RP, RA_MAX)
```